The electromagnetic four-potential derived as a path integral

1985 J. Phys. A: Math. Gen. 18 L13

(http://iopscience.iop.org/0305-4470/18/1/003)

J. Phys. A: Math. Gen. 18 (1985) L13-L16. Printed in Great Britain

LETTER TO THE EDITOR

The electromagnetic four-potential derived as a path integral

J R Ellis

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton BN1 9QH, UK

Received 19 September 1984

Abstract. An expression for the four-potential of an em field is derived as a path integral involving the fields, the formula being analogous to one given in elementary vector analysis.

When the vector field $\boldsymbol{F}(\boldsymbol{r})$ is solenoidal in a region R ($\operatorname{div}\boldsymbol{F}(\boldsymbol{r})=0$ in R), provided $\boldsymbol{F}(\boldsymbol{r})$ is continuously differentiable in R, a vector potential $\boldsymbol{G}(\boldsymbol{r})$ exists for $\boldsymbol{F}(\boldsymbol{r})$ ($\boldsymbol{F}(\boldsymbol{r})=\operatorname{curl}\boldsymbol{G}(\boldsymbol{r})$) given by

$$\boldsymbol{G}(\boldsymbol{r})=\int_0^1 t\boldsymbol{F}(t\boldsymbol{r})\wedge\boldsymbol{r}\,\mathrm{d}t, \tag{1}$$

to which may be added the gradient of any arbitrary scalar field ($\boldsymbol{F}(t\boldsymbol{r})\wedge\boldsymbol{r}\not\equiv\boldsymbol{0}$). The formula (1) was derived by Liebmann (1908) and remained unnoticed in the literature until it was rederived by Brand (1950). The method of its derivation based on a 'cone construction' is also given by Spain (1965), who cites the reference of Brand in his derivation. The formula (1) resembles the straight-path integral expression for the scalar potential for an irrotational field $\boldsymbol{F}(\boldsymbol{r})$,

$$\phi(\boldsymbol{r})=\int_0^1 \boldsymbol{F}(t\boldsymbol{r})\cdot\boldsymbol{r}\,\mathrm{d}t \tag{2}$$

($\operatorname{curl}\boldsymbol{F}(\boldsymbol{r})=\boldsymbol{0}$, $\boldsymbol{F}(\boldsymbol{r})=\operatorname{grad}\phi(\boldsymbol{r})$), though if one tries to extend (1) to curved paths in the same way as one extends (2), the generalisation is by no means clear. A generalisation of (1) to curved paths does exist. This is based on what might (for brevity) be termed a 'curved cone construction' where there exists a family of curves emanating from the origin, generating a surface shaped like a cone. This family replaces the family of straight lines which generate the cone in the usual derivation. In this letter we derive the generalised formula emphasising the applications of it to Maxwell's equations which appear not to have been noted before. For example, an expression for the four-potential A^μ as a path integral involving the fields appears not to have been noted before.

For the generalisation of (1) to curved paths we cite G N Ward, who has considered the detailed circumstances under which a vector potential exists for a solenoidal field using curvilinear coordinates (unpublished). The initial work used in this is applied to the present problem. We take a curved path from the origin O to the point P with position vector $\boldsymbol{r}_0$, describing this curve with parameter t ($0\leqslant t\leqslant 1$). For the construction of the generalised formula and for the construction of (1) it is necessary also to define a family of curves $\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{r},t)$, $0\leqslant t\leqslant 1$ of which the given curve is a member (the member $\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{r}_0, t)$, $0\leqslant t\leqslant 1$). Each curve of the family starts at O and ends at $\boldsymbol{r}$, so that $\boldsymbol{f}(\boldsymbol{r}, 0)=\boldsymbol{0}$ and $\boldsymbol{f}(\boldsymbol{r}, 1)=\boldsymbol{r}$ for each curve specified by $\boldsymbol{r}$. (It is assumed that $\boldsymbol{f}(\boldsymbol{r}, t)$ is twice continuously differentiable with respect to $\boldsymbol{r}$ and to t.) An example of such a function is $t\boldsymbol{r}$, and this specialisation will reduce the following argument to the proof of (1) using the usual cone construction referred to earlier. We consider the curved surface S_1 generated by the subfamily of curves $\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{r}(s), t)$ in which t varies continuously from 0 to 1 and $\boldsymbol{r}(s)$ is the position vector of a point on an arbitrary closed plane curve Γ; this curve is to lie in a fixed plane through P (P having position vector $\boldsymbol{r}_0$) and to enclose P. The parameter s is the distance parameter measured along Γ from some fixed point on it in a right-handed sense relative to the outward-drawn normal to the plane base (formed by Γ) of the cone-shaped body. This base we shall denote by S_2 (see figure 1). For the purposes of the calculation, Γ is taken infinitesimally close to P and the curves $\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{r}(s), t)$ likewise are taken close to the given curve. The given curve is ultimately the path along which the integration analogous to (1) is performed, and this situation is reached by a limiting process in which Γ contracts to P (while maintaining its fixed inclination) and the surface S_1 contracts to the curve OP. Before this limit is taken, Gauss's theorem applies to the (infinitesimal) cone-shaped domain bounded by S_1 and S_2 and under the circumstances we have described, the normal surface element on S_1 is given by

$$\mathrm{d}\boldsymbol{S}=\frac{\partial \boldsymbol{R}}{\partial s}\wedge\frac{\partial \boldsymbol{R}}{\partial t}\,\mathrm{d}s\,\mathrm{d}t.$$

Now using component notation $X_i=f_i(x_j(s), t)$, $i, j=1, 2, 3$, for $\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{r}(s), t)$,

$$\begin{aligned}(\mathrm{d}\boldsymbol{S})_i &= \varepsilon_{ijk}\frac{\partial X_j}{\partial s}\frac{\partial X_k}{\partial t}\,\mathrm{d}s\,\mathrm{d}t\\ &= \varepsilon_{ijk}\frac{\mathrm{d}x_m(s)}{\mathrm{d}s}\frac{\partial f_j}{\partial x_m(s)}\frac{\partial f_k}{\partial t}\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

(we omit the arguments of $\boldsymbol{f}$ and in future suppress suffixes in arguments when they are given). We have

$$\begin{aligned}\iint_{S_1}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} &= \iint_{S_1} F_i(f)\,\varepsilon_{ijk}\frac{\mathrm{d}x_m(s)}{\mathrm{d}s}\frac{\partial f_j}{\partial x_m(s)}\frac{\partial f_k}{\partial t}\,\mathrm{d}s\,\mathrm{d}t\\ &= -\oint_\Gamma\left\{\int_0^1 \varepsilon_{ijk}\frac{\partial f_i}{\partial x_m}F_j(f)\frac{\partial f_k}{\partial t}\,\mathrm{d}t\right\}\mathrm{d}x_m,\end{aligned}$$

and

$$\iint_{S_2}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}=\oint_\Gamma G_m(x)\,\mathrm{d}x_m,$$

using Stokes's theorem as in the usual cone construction, where $\boldsymbol{G}(\boldsymbol{r})$ is the vector potential for $\boldsymbol{F}(\boldsymbol{r})$. Applying Gauss's theorem to the domain S_1+S_2 and taking the limit as described earlier (noting the fixed inclination of S_2 is arbitrary),

$$G_m(x)=\int_0^1\frac{\partial f_i}{\partial x_m}\varepsilon_{ijk}F_j(f)\frac{\partial f_k}{\partial t}\,\mathrm{d}t \tag{3}$$

holds at $\boldsymbol{r}=\boldsymbol{r}_0$ ($x_i=x_{0i}$), where f_i denotes $f_i(x, t)$. The parameter s no longer appears.

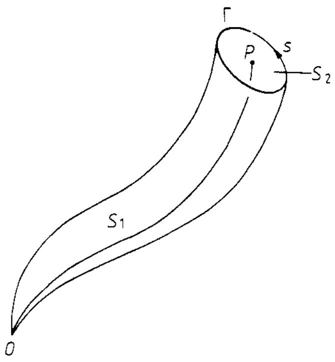

Figure 1.

k4Also since $\boldsymbol{r}_0$ is arbitrary within the space of the arguments of f, formula (3) applies generally for any $\boldsymbol{r}$ and is valid to the extent of an added gradient.

For the generalisation of (3) to Maxwell's equations we use relativistic notation (Greek suffixes range from 0 to 3). The metric of special relativity is taken in the form $c^2\,\mathrm{d}\tau^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ with $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $x^0 = ct$; we define duality $(*)$ in the ordinary way. Maxwell's equations are taken in the form $F^{\mu\nu}{}_{,\nu} = j^\mu$, $F^{*\mu\nu}{}_{,\nu} = 0$ and the latter equation states that $F^{*\mu\nu}$ is four-solenoidal. With the aid of the preceding work we deduce that $F^{*\mu\nu}$ is a four-dimensional curl:

$$F^{*\mu\nu}(x) = \varepsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta(x) \tag{4}$$

where the four-potential $A^\alpha(x)$ is given by

$$A_\alpha(x) = \int_0^1 (\partial_\alpha x'^\mu)\varepsilon_{\mu\nu\sigma\gamma}F^{*\nu\sigma}(x')\frac{\partial x'^\gamma}{\partial t}\,\mathrm{d}t = \int_O^P (\partial_\alpha x'^\mu)F_{\mu\gamma}(x')\,\mathrm{d}x'^\gamma \tag{5}$$

(to within an added gradient), the variable x' taking the role of the previous f. The integration in (5) is assumed performed along a world-line (or along a spacelike curve) from $(0, 0, 0, 0)$ to (x^μ). The parameter specifying the event on the path is t $(0 \leqslant t \leqslant 1)$ and the path itself is given by the equations $x'^\mu = x'^\mu(x, t)$ where (x^μ) is the end-point. By varying (x^μ) the family of such paths is represented. In the case where these paths are straight lines, given by

$$x'^\mu = tx^\mu, \qquad 0 \leqslant t \leqslant 1,$$

(x^μ) being an event in the future, the parameter t is proportional to the proper time measured along the world-line from O to (x^μ), and formula (5) then reduces to

$$A_\alpha(x) = \int_0^1 tF_{\alpha\gamma}(tx)x^\gamma\,\mathrm{d}t, \tag{6}$$

this formula being fully analogous to (1). The expression (6) may be verified directly. We adopt the method of Brand's vector analytic verification of (1) (Brand (1950) pp 163–4):

$$A_{\alpha,\beta}(x) = \int_0^1 t^2 F_{\alpha\gamma,\beta}(tx)x^\gamma\,\mathrm{d}t + \int_0^1 tF_{\alpha\beta}(tx)\,\mathrm{d}t,$$

$$\begin{aligned} A_{\alpha,\beta}(x)-A_{\beta,\alpha}(x) &= \int_0^1 t^2(F_{\alpha\gamma,\beta}(tx)+F_{\gamma\beta,\alpha}(tx))x^\gamma\,\mathrm{d}t+\int_0^1 2tF_{\alpha\beta}(tx)\,\mathrm{d}t \\ &= \int_0^1 t^2F_{\alpha\beta,\gamma}(tx)x^\gamma\,\mathrm{d}t+\int_0^1 2tF_{\alpha\beta}(tx)\,\mathrm{d}t \\ &= \int_0^1 (\partial/\partial t)(t^2F_{\alpha\beta}(tx))\,\mathrm{d}t \\ &= F_{\alpha\beta}(x). \end{aligned}$$

(The equivalent form of Maxwell's equations $F^{*\mu\nu}{}_{,\nu}=0$ and the identity

$$(\partial/\partial t)F_{\alpha\beta}(tx)=F_{\alpha\beta,\gamma}(tx)(\partial/\partial t)(tx^\gamma)$$

have been used.) The formula (6) automatically satisfies the Lorentz condition $A_{\alpha,}{}^{\alpha}=0$ when the integration is performed along a path in a source-free region of spacetime—it is assumed in (6) that $F^{\mu\nu}$ satisfies the further equation $F^{\mu\nu}{}_{,\nu}=j^\mu$. (We note that there is a similarity to the formally covariant equations for A_s^μ based on the Coulomb gauge (Rohrlich 1965).)

Finally we note that the form (3) may be used to find the representation of A_α in terms of a tensor potential. From the Lorentz condition $A_{\alpha,}{}^{\alpha}=0$ (four-solenoidal in the above sense) we deduce

$$A_\alpha(x)=(1/2!)\varepsilon_{\alpha\beta\gamma\sigma}\partial^\beta T^{\gamma\sigma}(x)$$

where $T^{\alpha\beta}$ is antisymmetric. Thus $A_\alpha(x)=\partial^\beta T^*_{\alpha\beta}(x)$, and $T^*_{\alpha\beta}$ represents the Hertz tensor potential of the em field whose components are the Hertz vectors $((T^*_{01}, T^*_{02}, T^*_{03})=\mathbf{\Pi}$, $(T^*_{23}\ T^*_{31}, T^*_{12})=c\mathbf{\Gamma})$. A path integral form for either $T^{\alpha\beta}$ or $T^*_{\alpha\beta}$ may be deduced directly from (3).

There may be possible applications to conservation laws and one might not rule out the application of (3) to relativistic quantum (field) theory.

References

Brand L 1950 *Am. Math. Mon.* **57** 161

Liebmann H 1908 *Leipziger Berichte* Bd. **60** 176

Rohrlich F 1965 *Classical Charged Particles* (Reading, Mass.: Addison-Wesley) pp 73–4

Spain B 1965 *Vector Analysis* (London: Van Nostrand) p 80